Let $\{a_n\}_{n\geq 1}$ be the sequence of real numbers defined by $a_1 = 3, a_2 = 5$ and for all $n \geq 2, a_{n+1} = \dfrac{1}{2}\left(a_n^2 + 1\right)$. Prove that

$$1 + 2\left(\sum_{k=1}^{n}\sqrt{\frac{F_k}{1+a_k}}\right)^2 < F_{n+2},$$

where F_n represents the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

- **5253:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate
  $$\int_0^1 \int_0^1 \frac{\ln x \cdot \ln(xy)}{1-xy} dxdy.$$

Solutions

- **5230:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive numbers x, y, z such that

$$\begin{aligned} x^2 + xy + \frac{y^2}{3} &= 41, \\ \frac{y^2}{3} + z^2 &= 16, \\ x^2 + xz + z^2 &= 25. \end{aligned}$$

Find the value of $xy + 2yz + 3xz$.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

Note that the given system is equivalent to

$$\begin{aligned} x^2 - 2x\frac{y}{\sqrt{3}}\cos 150^\circ + \left(\frac{y}{\sqrt{3}}\right)^2 &= \left(\sqrt{41}\right)^2, \\ \left(\frac{y}{\sqrt{3}}^2\right) + z^2 &= 4^2, \\ x^2 + 2xz\cos 120^\circ + z^2 &= 5^2. \end{aligned}$$

Let us take the right triangle ABC with $\angle B = 90^\circ, AB = 4$ and $BC = 5$ and let P be the interior point of ABC obtained as the intersection of the semicircle with diameter AB and the spanning arc of angle 120° (this is the locus of the points from which the segment BC is seen from an angle of 120°. Note that $\angle APB = 90^\circ,\ \angle BPC = 120^\circ$ and $\angle CPA = 150^\circ$. If we denote $x = CP, y = \sqrt{3}AP, z = BP$, we obtain the equations in the given system by applying the law of cosines to triangles $ACP,\ ABP$, and BCP.

Denoting the area of a triangle by $[\cdots]$ we have:

$$[ACP]+[ABP]+[BCP]=[ABC], \text{ or equivalently,}$$

$$\left(\frac{1}{2}\cdot PC\cdot PA\sin 150^\circ\right)+\left(\frac{1}{2}\cdot PA\cdot PB\right)+\left(\frac{1}{2}\cdot PC\cdot PB\cdot\sin 120\right)=\frac{1}{2}\cdot AB\cdot BC.$$

That is,

$$\left(\frac{1}{2}\cdot x\cdot\frac{y}{\sqrt{3}}\cdot\frac{1}{2}\right)+\left(\frac{1}{2}\cdot\frac{y}{\sqrt{3}}z\right)+\left(\frac{1}{2}\cdot x\cdot z\cdot\frac{\sqrt{3}}{2}\right)=\frac{1}{2}\cdot 4\cdot 5.$$

Multiplying by $4\sqrt{3}$, gives us that

$$xy+2yz+3xz=40\sqrt{3}.$$

Comment by Bruno: Very similar problems to this one are problems #12 of the 1984 All-Soviet Union Mathematical Olympiad and problem # E1 in *Problem Solving Strategies* by Arthur Engel (Springer-Verlag), 1998, pp. 380-381

Solution 2 by Arkady Alt, San Jose, California, USA

Let $S=xy+2yz+3xz$. By replacing y in the original problem with $y\sqrt{3}$ we obtain:

$$\begin{aligned}
x^2+xy\sqrt{3}+y^2 &= a^2+b^2,\\
y^2+z^2 &= a^2, \text{ and}\\
x^2+xz+z^2 &= b^2, \text{where } a=4,\ b=5, \text{ and}\\
S &= xy\sqrt{3}+2\sqrt{3}yz+3xz, \text{ or}\\
x^2+y^2-2\cos\frac{5\pi}{6}xy &= a^2+b^2,\\
y^2+z^2-2\cos\frac{\pi}{2}yz &= a^2,\\
x^2+z^2-2\cos\frac{2\pi}{3}xz &= b^2,\\
\frac{S}{2\sqrt{3}} &= xy\sin\frac{5\pi}{6}+yz\sin\frac{\pi}{2}+zx\sin\frac{2\pi}{3}.
\end{aligned}$$

Consider four points A,B,C,P on a plane such that $PA=x, PB=y, PM=z$ and $\angle APB=\frac{5\pi}{6}, \angle BPC=\frac{\pi}{2}, \angle CPA=\frac{2\pi}{3}$.

Since $\frac{5\pi}{6}+\frac{2\pi}{3}+\frac{\pi}{2}=2\pi$ then, accordingly to the equalities

$$x^2+y^2-2\cos\frac{5\pi}{6}xy = a^2+b^2,$$

$$\begin{aligned} y^2+z^2-2\cos\frac{\pi}{2}yz &= a^2, \\ x^2+z^2-2\cos\frac{2\pi}{3}xz &= b^2, \text{where} \end{aligned}$$

P is the interior point of the right triangle ABC with right angle at C, and sides $BC=a,\ AC=b$.

Then we have $[ABC]=[APB]+[BPC]+[CPA] \iff$

$$\frac{AC\cdot BC}{2}=\frac{PA\cdot PB}{2}\sin\frac{5\pi}{6}+\frac{PB\cdot PC}{2}\sin\frac{\pi}{2}+\frac{PC\cdot PA}{2}\sin\frac{2\pi}{3} \iff$$

$$\begin{aligned} a\cdot b &= xy\sin\frac{5\pi}{6}+yz\sin\frac{\pi}{2}+zx\sin\frac{2\pi}{3} \iff \\ ab &= \frac{S}{2\sqrt{3}} \iff S=2\sqrt{3}ab. \end{aligned}$$

For $a=4$ and $b=5$ we obtain $S=40\sqrt{3}$.

Remark: The original problem is a particular case of a more general problem.

Given positive numbers $x,\ y,\ z,\alpha,\beta,\gamma,a,b,c$ such that $\alpha+\beta+\gamma=2\pi,\ a,b,c$ and

$$\begin{cases} x^2+y^2-2\cos\gamma xy=c^2 \\ y^2+z^2-2\cos\alpha yz=a^2 \\ x^2+z^2-2\cos\beta xz=b^2. \end{cases}$$

Find the value of $|xy\sin\gamma+yz\sin\alpha+zx\sin\beta|$. This problem has a simple vector interpretation.

Indeed, let $\mathbf{x},\mathbf{y},\mathbf{z}$ be three pairwise non-collinear vectors on a plane such that

$$\|\mathbf{x}\|=x,\ \|\mathbf{y}\|=y,\ \|\mathbf{z}\|=z$$

the oriented angles between the pairs of vectors are

$$\angle(\mathbf{x},\mathbf{y})=\gamma, \angle(\mathbf{y},\mathbf{z})=\alpha, \angle(\mathbf{z},\mathbf{x})=\beta.$$

Then according to the conditions of problem, we also have

$$\begin{aligned} \|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y}) \\ &= \|\mathbf{x}\|^2-2(\mathbf{x}\cdot\mathbf{y})+\|\mathbf{y}\|^2 \\ &= x^2+y^2-2\cos\gamma\cdot xy \\ &= c^2 \text{ and similarly,} \\ \|\mathbf{y}-\mathbf{z}\|^2 &= a^2, \\ \|\mathbf{z}-\mathbf{x}\|^2 &= b^2. \end{aligned}$$

It is easy to see that

$$a+b=\|\mathbf{y}-\mathbf{z}\|+\|\mathbf{z}-\mathbf{x}\|\geq\|\mathbf{x}-\mathbf{y}\|=c,$$

and since $\mathbf{y}-\mathbf{z}$ and $\mathbf{z}-\mathbf{x}$ aren't collinear then $a+b>c$.

Similarly, $b+c>a$ and $c+a>b$. Thus the positive numbers a,b,c define a triangle with area with semi-perimeter s and area $F=\sqrt{s(s-a)(s-b)(s-c)}$.

Definition

For any two vectors $\mathbf{x}=(x_1,x_2)$, $\mathbf{y}=(y_1,y_2)$ we define the "exterior product" of two vectors in the plane as follows:

$$\mathbf{x}\wedge\mathbf{y}=x_1y_2-x_2y_1.$$

From this definition we can immediately obtain the following properties of the exterior product:

- 1. $\mathbf{x}\wedge\mathbf{y}=-\mathbf{y}\wedge\mathbf{x}$,
- **2.** $\mathbf{x}\wedge\mathbf{x}=\mathbf{0}$,
- **3.** $\mathbf{x}\wedge(\mathbf{y}+\mathbf{z})=\mathbf{x}\wedge\mathbf{y}+\mathbf{x}\wedge\mathbf{z}$ and $(\mathbf{x}+\mathbf{y})\wedge\mathbf{z}=\mathbf{x}\wedge\mathbf{z}+\mathbf{y}\wedge\mathbf{z}$,
- **4.** $(k\,\mathbf{x})\wedge\mathbf{y}=\mathbf{x}\wedge k\mathbf{y}=k(\mathbf{x}\wedge\mathbf{y})$.

One more property expresses the geometric essence of the exterior product in a plane. Let

$$\mathbf{e}=(0,1),\varphi=\angle(\mathbf{e},\mathbf{x}),\psi=\angle(\mathbf{e},\mathbf{y}),\ \angle(\mathbf{x},\mathbf{y})=\psi-\varphi$$

and since

$$\begin{aligned}
(x_1,x_2) &= \|\mathbf{x}\|(\cos\varphi,\sin\varphi),\\
(y_1,y_2) &= \|\mathbf{y}\|(\cos\psi,\sin\psi),\ \text{then}\\
\mathbf{x}\wedge\mathbf{y} &= x_1y_2-x_2y_1\\
&= \|\mathbf{x}\|\,\|\mathbf{y}\|(\cos\varphi\sin\psi-\sin\varphi\cos\psi)\\
&= \|\mathbf{x}\|\,\|\mathbf{y}\|\sin\angle(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Hence, $\mathbf{x}\wedge\mathbf{y}$ is the oriented area of the parallelogram defined by $(\mathbf{x},\mathbf{y})$, and $|\mathbf{x}\wedge\mathbf{y}|$ is area of this parallelogram.

Coming back to our problem we obtain

$$\begin{aligned}
xy\sin\gamma+yz\sin\alpha+zx\sin\beta &= \|\mathbf{x}\|\,\|\mathbf{y}\|\sin\angle(\mathbf{x},\mathbf{y})+\|\mathbf{y}\|\,\|\mathbf{z}\|\sin\angle(\mathbf{y},\mathbf{z})+\|\mathbf{z}\|\,\|\mathbf{x}\|\sin\angle(\mathbf{z},\mathbf{x})\\
&= \mathbf{x}\wedge\mathbf{y}+\mathbf{y}\wedge\mathbf{z}+\mathbf{z}\wedge\mathbf{x}.
\end{aligned}$$

Using properties $1-4$ we have

$$(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})=\mathbf{x}\wedge\mathbf{x}-\mathbf{y}\wedge\mathbf{x}-\mathbf{x}\wedge\mathbf{z}+\mathbf{y}\wedge\mathbf{z}=\mathbf{x}\wedge\mathbf{y}+\mathbf{z}\wedge\mathbf{x}+\mathbf{y}\wedge\mathbf{z}.$$

Thus,

$$|xy\sin\gamma + yz\sin\alpha + zx\sin\beta| = |(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})| \text{ and since}$$

$|(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})|$ is the area of the parallelogram defined by vectors $\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{z}$ which is equal to $2F$. So, we obtain finally that

$$|xy\sin\gamma + yz\sin\alpha + zx\sin\beta| = 2\sqrt{s(s-a)(s-b)(s-c)}.$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $xy + 2yz + 3xz = 40\sqrt{3}$.

Denote the given equations by (1), (2), and (3) in given order. Then (2) + (3) − (1) gives $2z^2 + xz - xy = 0$, so that

$$y = \frac{2z^2}{x} + z. \qquad (4)$$

Substitute y of (4) into (2) and simplifying gives

$$z^4 + xz^3 + x^2z^2 = 12x^2. \qquad (5)$$

From (5) and (3) we have

$$z^2 = \frac{12x^2}{25}. \qquad (6)$$

Substitute z^2 of (6) into (3) and simplifying, we obtain

$$z = \frac{625 - 37x^2}{25x}. \qquad (7)$$

Substitute z of (7) into (6) and simplifying, we obtain

$$1069x^4 - 46250x^2 + 390625 = 0. \qquad (8)$$

Now (8) gives

$$x^2 = \frac{625\left(37 - 10\sqrt{3}\right)}{1069}, \text{ and } \frac{625\left(37 + 10\sqrt{3}\right)}{1069}.$$

If $x^2 = \dfrac{625\left(37 + 10\sqrt{3}\right)}{1069}$, then by (6), we have $z^2 = \dfrac{300\left(37 + 10\sqrt{3}\right)}{1069}$. Then using (3), we see that $xz = -\dfrac{250\left(30 + 37\sqrt{3}\right)}{1069} < 0$, must be rejected. Hence by (6) and (2), we have

$$x^2 = \frac{625\left(37 - 10\sqrt{3}\right)}{1069},\ z^2 = \frac{300\left(37 - 10\sqrt{3}\right)}{1069},\ y^2 = \frac{12\left(1501 + 750\sqrt{3}\right)}{1069}.$$

By (1) and (3) we obtain

$$xy = \frac{50\left(294 + 65\sqrt{3}\right)}{1069},\ xz = \frac{250\left(-30 + 37\sqrt{3}\right)}{1069}.$$